Que :- Cosider the object in the figure. Calculate for

(i) the length BC

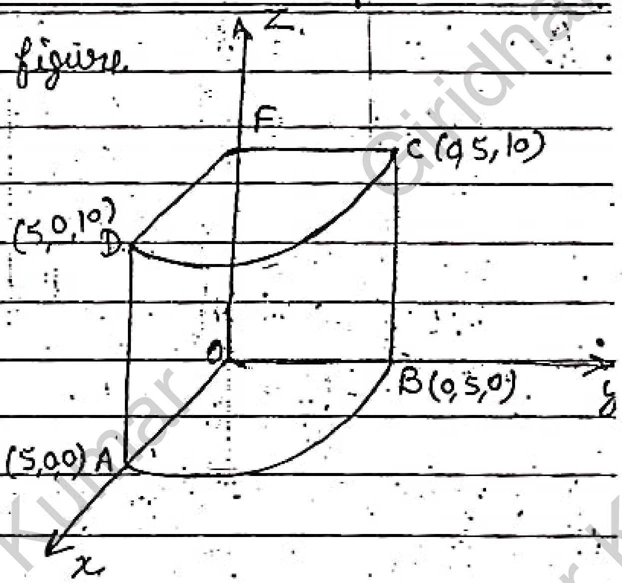
(ii) the length CD

(iii) Surface area ABCD

(iv) Surface area ABO

(V) Surface area AOFD

(vi) Volume ABDCFO



(i) $BC = \int_0^{10} dz$

$= 10$

(ii) $CD = \int \rho\, d\phi$ $\quad [\rho = 5]$

$= 5\int_0^{\pi/2} d\phi$

$= \frac{5\pi}{2}$

(iii) $ABCD = \int_s \rho\, d\phi\, dz$

$= 5\int_0^{\pi/2} d\phi \int_0^{10} dz = 5\left[\frac{\pi}{2}\right][10]$

$= 25\pi$

z-dim cross-sec

(iv) $ABO = \int_s \rho\, d\rho\, d\phi$

$= \int_0^5 \rho\, d\rho \int_0^{\pi/2} d\phi = \left[\frac{\rho^2}{2}\right]_0^5 \left[\phi\right]_0^{\pi/2}$

$= \frac{25}{2}\cdot\frac{\pi}{2} = \frac{25\pi}{4}$

(V) $\underset{(\text{area})}{AOFD} = \int_s d\rho\, dz$ $\quad$ area vector $\hat{\phi}$

$= \int_0^5 d\rho \int_0^{10} dz = (5)(10) = 50$

(vi) $\underset{(\text{Volume})}{ABDCFO} = \int_V \rho\, d\rho\, d\phi\, dz = \int_0^5 \rho\, d\rho \int_0^{\pi/2} d\phi \int_0^{10} dz$

$= \frac{25}{2}\cdot\frac{\pi}{2}\cdot 10 = \frac{125\pi}{2}$

Q.1:- (a) $f(x,y,z) = x^2 + y^3 + z^4$

$$\cancel{\text{grad } f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2x\,\hat{i} + 3y^2\,\hat{j} + 4z^3\,\hat{k}$$

(b) $f(x,y,z) = x^2y^3z^4$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

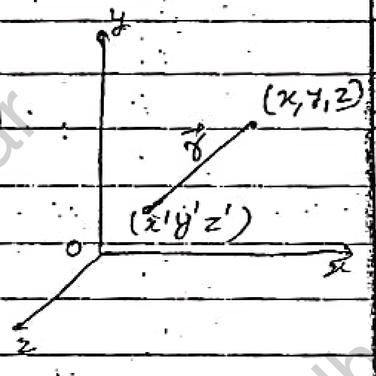
$$= 2xy^3z^4\,\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^3\,\hat{k}$$

$$= xyz\left(2y^2z^3\,\hat{i} + 3xyz^3\,\hat{j} + 4xy^2z^2\,\hat{k}\right)$$

(c) $f(x,y,z) = e^x \sin y \ln(z)$

$$\nabla f = e^x \sin y \ln(z)\,\hat{i} + e^x \cos y \ln(z)\,\hat{j} + \frac{e^x \sin y}{z}\,\hat{k}$$

Q.2 (a) $\vec{\nabla}(r^2) = 2\vec{r}$



$$\vec{\nabla}(r^2) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(x^2+y^2+z^2\right)$$

$$= 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$= 2\,(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 2\vec{r}$$

(b) $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$

$$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(x^2+y^2+z^2\right)^{-1/2}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\,2x\,\hat{i} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2}\,2y\,\hat{j}$$

$$- \frac{1}{2}(x^2+y^2+z^2)^{-3/2}\,2z\,\hat{k}$$

$$= -(x^2+y^2+z^2)^{-3/2}\,(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(\sqrt{x^2+y^2+z^2})^3} = -\frac{\vec{r}}{r^3} = \frac{-r\hat{r}}{r^3} = \frac{-\hat{r}}{r^2}$$

(c) $\nabla(r^n) = n r^{n-2}\vec{r}$

$$\nabla(r^n) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)^{n/2}$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x)\hat{i} + \cdots$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= n\left[(x^2+y^2+z^2)^{1/2}\right]^{n-2}\vec{r}$$

$$= n r^{n-2}\vec{r}$$

Q.3: (a) $\vec{v_a} = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

$$\text{div}\, v_a = \vec{\nabla}\cdot\vec{v_a} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}\right)$$

$$= 2x + - 2x = 0$$

(b) $\vec{v_b} = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$

$$\vec{\nabla}\cdot\vec{v_b} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}\right)$$

$$= y + 2z + 3x$$

(c) $\vec{v_c} = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$

$$\vec{\nabla}\cdot\vec{v_c} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(y^2\hat{x} + (2xy+z^2)\hat{y} + 2yz\hat{z}\right)$$

$$= 2x + 2y = 2(x+y)$$

Q.5: (a) $T_a = x^2 + 2xy + 3z + 4$

$$\nabla^2 T_a = \left[\frac{\partial^2}{\partial x^2}(x^2+2xy+3z+4) + \frac{\partial^2}{\partial y^2}(x^2+2xy+3z+4) + \frac{\partial^2}{\partial z^2}(x^2+2xy+3z+4)\right]$$

$$= \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial z}(3)$$

$$= 2$$

(b) $T_b = \sin x \sin y \sin z$

$$\nabla^2 T_b = \frac{\partial^2}{\partial x^2}(\sin x \sin y \sin z) + \frac{\partial^2}{\partial y^2}(\sin x \sin y \sin z) + \frac{\partial^2}{\partial z^2}(\sin x \sin y \sin z)$$

$$= \frac{\partial}{\partial x}(\cos x \sin y \sin z) + \frac{\partial}{\partial y}(\sin x \cos y \sin z) + \frac{\partial}{\partial z}(\sin x \sin y \cos z)$$

$$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$$

$$= \sin x \sin y \sin z(-3)$$

$$= -3 \sin x \sin y \sin z$$

(c) $T_c = e^{-5x} \sin 4y \cos 3z$

$$\nabla^2 T_c = \frac{\partial^2}{\partial x^2}\left[e^{-5x} \sin 4y \cos 3z\right] + \frac{\partial^2}{\partial y^2}\left(e^{-5x} \sin 4y \cos 3z\right) + \frac{\partial^2}{\partial z^2}\left(e^{-5x} \sin 4y \cos 3z\right)$$

$$= 25 e^{-5x} \sin 4y \cos 3z \cancel{+} -16 e^{-5x} \sin 4y \cos 3z - 9 e^{-5x} \sin 4y \cancel{\sin 3z} \cos 3z$$

$$= e^{-5x} \sin 4y \cos 3z \left[25 - 16 - 9\right]$$

$$= 0$$

(d) $\vec{v} = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$

$$\nabla^2 \vec{v} = \left(\frac{\partial^2}{\partial x^2} x^2 + \frac{\partial^2}{\partial y^2} x^2 + \frac{\partial^2}{\partial z^2} x^2\right)\hat{x} + \left(\frac{\partial^2}{\partial x^2} 3xz^2 + \frac{\partial^2}{\partial y^2} 3xz^2 + \frac{\partial^2}{\partial z^2} 3xz^2\right)\hat{y}$$

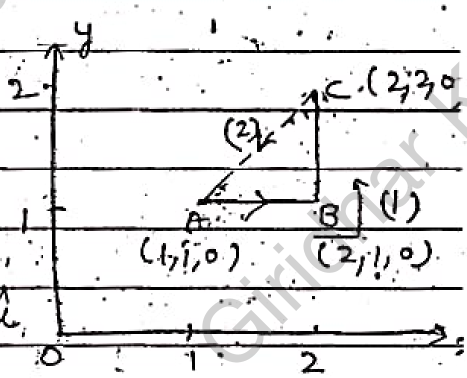
$$+\left(\frac{\partial^2}{\partial x^2}(-2xz) + \frac{\partial^2}{\partial y^2}(-2xz) + \frac{\partial^2}{\partial z^2}(-2xz)\right)\hat{z}$$

$$\nabla^2\vec{v} = 2\hat{x} + 6x\hat{y} + 0 = 2\hat{x} + 6x\hat{y}$$

Q.6 $\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$

$a = (1,1,0)$, $b = (2,2,0)$

$$\oint \vec{v}\cdot d\vec{l} = \int_{AB} \vec{v}\cdot d\vec{l} + \int_{BC} \vec{v}\cdot d\vec{l} + \int_{CA} \vec{v}\cdot d\vec{l}$$



$$= \int_1^2 \left(y^2\hat{x} + 2x(y+1)\hat{y}\right)\cdot dx\,\hat{x}$$

$$+ \int_1^2 \left(y^2\hat{x} + 2x(y+1)\hat{y}\right)\cdot dy\,\hat{y}$$

$$+ \int_2^1 \left(y^2\hat{x} + (2x(y+1)\hat{y}\right)\cdot dy\,\hat{y}$$

$$= \int_1^2 y^2\,dx + \int_1^2 2x(y+1)\,dy + \int_2^1 (2x(y+1))\,dy$$

$$= y^2[x]_1^2 + 2x\left[\frac{y^2}{2} + y\right]_1^2 = y^2(2-1) + 2x\left[\frac{4}{2} + 2 - \frac{1}{2} - 1\right] + 2x\left[\frac{y^2}{2} + y\right]_2^1$$

$$= y^2 + 2x\left(\frac{5}{2}\right) = y^2 + 5x - 5x = 1 + 10 - 5 = 6$$

Q.7 ~~$v = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$~~

$$\oint_S \vec{v}\cdot d\vec{s} =$$